Hjemmeøving 13, TMA4115, T. Alexander Lystad

Oppgave 4.3.10

$y_1' = y_1 + y_2,\ y_1(0) = 1$

$y_2' = 4y_1 + y_2,\ \ y_2(0) = 6$

Coefficient matrix: $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = -1$

Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$y(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$

$y_1(t) = c_1 e^{3t} - c_2 e^{-t}$

$y_2(t) = 2c_1 e^{3t} + 2c_2 e^{-t}$

$y_1(0) = c_1 - c_2 = 1$

$y_2(0) = 2c_1 + 2c_2 = 6$

$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & 4 \end{bmatrix}$

$4c_2 = 4 \Rightarrow c_2 = 1$

$c_1 - c_2 = 1 \Rightarrow c_1 = 1 + c_2 = 1 + 1 = 2$

$y_1(t) = 2e^{3t} - e^{-t}$

$y_2(t) = 4e^{3t} + 2e^{-t}$

Oppgave 4.3.18

Oppgave 6.4.13

$$A=\begin{bmatrix}4&1&1\\1&4&1\\1&1&4\end{bmatrix}$$

$$\left|A-\lambda I\right|=0$$

$$\begin{vmatrix}4-\lambda&1&1\\1&4-\lambda&1\\1&1&4-\lambda\end{vmatrix}=-\left(-6+\lambda\right)\left(-3+\lambda\right)^2=0\Rightarrow \underline{\lambda_1=6,\lambda_2=3}$$

$$\left(A-\lambda_1 I\right)v=0$$

$$\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2&1&1\\0&-\frac{3}{2}&\frac{3}{2}\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$z=s$$

$$-\tfrac{3}{2}y+\tfrac{3}{2}z=0\Rightarrow y=z=s$$

$$-2x+y+z=-2x+s+s=0\Rightarrow x=s$$

$$\underline{v_1=\left(1,1,1\right)}$$

$$\left(A-\lambda_2 I\right)v=0$$

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x+y+z=0\Rightarrow z=-x-y$$

$$\left(x,y\right)=\left(0,1\right)\Rightarrow z=-1$$

$$\left(x,y\right)=\left(1,0\right)\Rightarrow z=-1$$

$$\underline{v_2=\left(0,1,-1\right),v_3=\left(1,0,-1\right)}$$

$$\underline{\underline{u_1=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{3}\right)}}$$

$$\underline{\underline{u_2=\left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)}}$$

$$\underline{\underline{u_3=\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)}}$$

Oppgave 6.4.25

En matrise M er ortogonal hvis $MM^T = I$.

AB er ortogonal hvis $AB(AB)^T = I$.

$AB(AB)^T = ABB^T A^T = AIA^T = AA^T = I$

$\underline{\underline{AB \text{ er ortogonal.}}}$

Oppgave 8.1.1

$f(x) = 2x^2 + y^2 - 8x - 6y + 13$

$f(x) = 2(x^2 - 4x) + (y^2 - 6y) + 13$

$f(x) = 2(x^2 - 4x + 4) + (y^2 - 6y + 9) + 13 - 4 - 9$

$f(x) = 2(x-2)^2 + (y-3)^2$

$x' = x - 2, y' = y - 3$

$f(x) = 2x' + y'$

$f(x)$ er en ellipse og det nye x'y'-koordinatsystemet har

origo$(2,3)$ i det gamle xy-koordinatsystemet.

Oppgave 8.1.11

$4x^2 + 6xy - 4y^2 = 5$

On matrix form: $x^T Ax + Bx + f = x^T \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} x - 5 = 0$

Eigenvalues of A: $\lambda_1 = -5, \lambda_2 = 5$

Eigenvectors of A: $v_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Unit eigenvectors of A: $u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}, u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$P = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

$P^T AP = \left(\frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \right) \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} \left(\frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \right) = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$

$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 5x'^2 - 5y'^2$

$(Px')^T APx' + f = 5x'^2 - 5y'^2 - 5 = x'^2 - y'^2 - 1$

$\Rightarrow \underline{\underline{x'^2 - y'^2 = 1;\ \text{hyperbel}}}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) = \underline{\underline{53.13°}}$

Oppgave 8.1.23

Eksamensoppgave A-33 a)

$$A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = (9-\lambda)(16-\lambda) - 12^2 = \lambda^2 - 25\lambda = 0$$

$$\lambda(\lambda - 25) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 25$$

$$(A - \lambda_1 I)v = 0$$

$$\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + \tfrac{4}{3}y = 0$$

$$x = -\tfrac{4}{3}y$$

$y = 0$ gir $x = -\frac{4}{3} \cdot 0 = 0$

$y = 1$ gir $x = -\frac{4}{3} \cdot 1 = -\frac{4}{3}$

$$v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)v = 0$$

$$\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x - \tfrac{3}{4}y = 0$$

$$x = \tfrac{3}{4}y$$

$y = 0$ gir $x = \frac{3}{4} \cdot 0 = 0$

$y = 1$ gir $x = \frac{3}{4} \cdot 1 = \frac{3}{4}$

$$v_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

$$u_1 = \frac{3}{5}\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, u_2 = \frac{4}{5}\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$P = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$P^{-1}AP = \left(\frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\right)\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\left(\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\right) = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} = D$$